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RESTRICTION OF THE GUICHARDET–WIGNER PSEUDOCHARACTER ON $\mathrm{SL}(2,\mathbb{R})$ TO A SUBGROUP COVERING $\mathrm{SL}(2,\mathbb{Z})$

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Abstract. It is proved that the restriction of the Guichardet–Wigner pseudocharacter on the universal covering group G of $\mathrm{SL}(2,\mathbb{R})$ to a subgroup L of G covering the group $\mathrm{SL}(2,\mathbb{Z})$ is a nontrivial pseudocharacter (rather than an ordinary character) of L.

§ 1. Introduction

Recall that any mapping f of a given group G into the group of reals such that

$$|f(g_1g_2) - f(g_1) - f(g_2)| \le C, \qquad g_1, g_2 \in G,$$

for some C is referred to as a *quasicharacter* on G and a quasicharacter on G is said to be a *pseudocharacter* on G if

$$f(g^n) = nf(g)$$

for any $g \in G$ and $n \in \mathbb{Z}$; see [1–3].

One of the most useful pseudocharacters is the so-called Guichardet–Wigner pseudocharacter on connected simply connected Hermitian symmetric semisimple Lie groups, see [3, 4]; the most well-studied Guichardet–Wigner pseudocharacter is the Guichardet–Wigner pseudocharacter defined

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on the universal covering group of $\mathrm{SL}(2,\mathbb{R})$ (or, equivalently, of $\mathrm{SU}(1,1)$), see, e.g., [5].

Recall one of the standard parametrizations of G. Following [6], we write out the elements of G by pairs,

$$G = ((\gamma,\omega) \mid \gamma \in \mathbb{C}, |\gamma| < 1, \omega \in \mathbb{R}),$$

where

$$(\gamma,\omega) = \left\{\omega, \begin{pmatrix} e^{i\omega}(1-|\gamma|^2)^{-1/2} & e^{-i\omega}\overline{\gamma}(1-|\gamma|^2)^{-1/2} \\ e^{i\omega}\gamma(1-|\gamma|^2)^{-1/2} & e^{-i\omega}(1-|\gamma|^2)^{-1/2} \end{pmatrix}\right. .$$

Let $\pi\colon G \to \mathrm{SU}(1,1)$ be the canonical covering mapping. Let L be the discrete subgroup of G generated by the elements

$$g_0 = \left(2\pi, \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\right),$$

$$g_1 = \left(\pi/2, \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}\right),$$

and

$$g_2 = \left(\arctan(1/2), \begin{pmatrix} 1+i/2 & -i/2 \\ i/2 & 1-i/2 \end{pmatrix}\right)$$

covering the subgroup corresponding to the subgroup $\mathrm{SL}(2,\mathbb{Z})$ of $\mathrm{SL}(2,\mathbb{R})$ in the realization $\mathrm{SU}(1,1)$.

In the present note we prove that the restriction of the Guichardet–Wigner pseudocharacter to L is a nontrivial pseudocharacter of L (rather than an ordinary character).

§ 2. Main results

Theorem 1. *The subgroup L is discrete in G.*

Proof. If L has an accumulation point in G, then either this point is an accumulation point of the corresponding fiber, which is impossible because this fiber is discrete, or an accumulation point of a family of points whose projections to the projection of L to the subgroup isomorphic to $\mathrm{SL}(2,\mathbb{Z})$, which is also impossible because $\mathrm{SL}(2,\mathbb{Z})$ is discrete.

Theorem 2. *The restriction of the Guichardet–Wigner pseudocharacter on the group G to L is a nontrivial pseudocharacter of the subgroup L (i.e., this restriction is not an ordinary real character of L).*

Proof. The value of the Guichardet–Wigner pseudocharacter χ on G at the point

$$g = \left(\omega, \begin{pmatrix} e^{i\omega} & 0 \\ 0 & e^{-i\omega} \end{pmatrix}\right) \in G$$

is equal to ω for every $\omega \in G$ by the very definition of the Guichardet–Wigner pseudocharacter [3, 4]; the value of the Guichardet–Wigner pseudocharacter χ on G at the point g_0 is thus equal to 2π. It can readily be seen by taking the elements g_1 and g_2 to some high positive integer power that $\chi(g_1) = \pi/2$ and $\chi(g_2) = 0$ (the former formula is obvious; the latter formula follows from the fact that the elements

$$g_t = \left(\arctan(t), \begin{pmatrix} 1+it & -it \\ it & 1-it \end{pmatrix}\right)$$

form a one-parameter subgroup of G with bounded coordinate $\arctan(t)$, which immediately implies that χ vanishes on this one-parameter subgroup). On the other hand, the product $g_2 g_1$ is an element of the form (ω_0, a), where

$$a^3 = E = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Therefore,

$$\omega_0 = 2k\pi$$

for an integer k. This implies that

$$\chi(g_2 g_1) = 2k\pi/3,$$

which cannot be equal to

$$\chi(g_2) + \chi(g_1) = \pi/2.$$

We finally see that the restriction of χ to L, which is obviously a pseudocharacter, is not an ordinary real character, as was to be proved.

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